Branching rules for the Weyl group orbits of the Lie algebra $A_n$



2009 J. Phys. A: Math. Theor. 42 485203

(http://iopscience.iop.org/1751-8121/42/48/485203)

J. Phys. A: Math. Theor. **42** (2009) 485203 (15pp) doi:10.1088/1751-8113/42/48/485203

# Branching rules for the Weyl group orbits of the Lie algebra $A_n$

**M Larouche[1], M Nesterenko[2] and J Patera[1]**

[1] Centre de Recherches Mathématiques, Université de Montréal, CP 6128 Centre-ville, Montréal, Québec, H3C 3J7, Canada
[2] Institute of Mathematics of NAS of Ukraine, 3, Tereshchenkivs'ka str, Kyiv-4, 04216, Ukraine

E-mail: larouche@dms.umontreal.ca, maryna@imath.kiev.ua and patera@crm.umontreal.ca

Received 14 September 2009, in final form 15 October 2009
Published 11 November 2009
Online at stacks.iop.org/JPhysA/42/485203

**Abstract**
The orbits of Weyl groups $W(A_n)$ of simple $A_n$-type Lie algebras are reduced to the union of orbits of the Weyl groups of maximal reductive subalgebras of $A_n$. Matrices transforming points of the orbits of $W(A_n)$ into points of subalgebra orbits are listed for all cases $n \leqslant 8$ and for the infinite series of algebra–subalgebra pairs $A_n \supset A_{n-k-1} \times A_k \times U_1$, $A_{2n} \supset B_n$, $A_{2n-1} \supset C_n$, $A_{2n-1} \supset D_n$. Numerous special cases and examples are shown.

PACS numbers: −02.20.Qs, 02.20.Sv
Mathematics Subject Classification: 20F55, 22E46

## 1. Introduction

Finite groups generated by reflections in an $n$-dimensional real Euclidean space $\mathbb{R}^n$ are commonly known as finite Coxeter groups [8, 9]. Finite Coxeter groups are split into two classes: crystallographic and non-crystallographic groups. Crystallographic groups are often referred to as Weyl groups of semisimple Lie groups or Lie algebras. They are symmetry groups of some lattices in $\mathbb{R}^n$. There are four infinite series (as to the admissible values of rank $n$) of such groups, and five isolated exceptional groups of ranks 2, 4, 6, 7 and 8. Non-crystallographic finite Coxeter groups are the symmetry groups of regular 2D polygons (the dihedral groups), and two exceptional groups, for $n = 3$ (the icosahedral group of order 120) and $n = 4$, which is of order $120^2$.

We consider the orbits of the Weyl groups $W(A_n)$ of the simple Lie algebras of type $A_n$, $n \geqslant 1$, equivalently the Weyl groups of the simple Lie group $SL(n+1, \mathbb{C})$, or of its compact real form $SU(n+1)$. The order of such a Weyl group is $(n+1)!$. An orbit of $W(A_n)$ is a set of distinct points in $\mathbb{R}^n$, obtained from a chosen single (seed) point, say $\lambda \in \mathbb{R}^n$, by the application of $W(A_n)$ to $\lambda$. Hence, an orbit $W_\lambda$ of $W(A_n)$ contains at most $(n+1)!$ points. The points of $W_\lambda$ are equidistant from the origin. It should be noted that the group $W(A_n)$ is isomorphic to the permutation group of $n + 1$ elements. Although we make no use of this fact here, it reveals a rather different perspective on our problem [18].

Geometrically, points of the same orbit can be seen as vertices of a convex polytope generated from $\lambda$. There is a method for counting and describing the faces of all dimensions of such polytopes in the real Euclidean space $\mathbb{R}^n$. It uses an easy recursive decoration of the corresponding Coxeter–Dynkin diagrams [3].

Weyl group orbits are closely related to weight systems of finite-dimensional irreducible representations of corresponding Lie algebras. More precisely, the weight system is a union of several Weyl group orbits. Which orbits are composed into a particular weight system is in principle known. An efficient algorithm for the computation exists [2]. The representations are finding innumerable applications in science. Very often, such applications can be carried through just by our knowledge of the corresponding weight system. It is conceivable that some of the applications would find interesting new possibilities when working with individual orbits only.

The list of possible reductions of $W(A_n)$ orbits is a result of a major classification problem solved more than half a century ago, when the maximal reductive subalgebras of simple Lie algebras, in particular of $A_n$, were determined [1, 4]. We exploit that classification without further reference to it.

In this paper, we consider orbits of $W(A_n)$ and their reduction to orbits of the Weyl groups of maximal reductive subalgebras of $A_n$. In the physics literature, a similar task [12] is often called computation of branching rules. We will consider two types of maximal reductive subalgebras, maximal reductive subalgebras that are not semisimple [1], and subalgebras that are maximal among reductive subalgebras, but which are in fact semisimple. Thus, the second type of subalgebras are obtained from the list of [4] by eliminating semisimple subalgebras that are part of the reductive subalgebras classified in [1].

The present paper can be understood as a continuation of [7], where the orbits are seen as elements of a ring of reflection-generated polytopes in $\mathbb{R}^n$. In that paper, the main problem was to reduce products of Weyl group orbits/polytopes into a sum of Weyl group orbits. Here, our problem is to transform/reduce/branch each polytope/orbit into a sum of concentric polytopes with lower symmetry, and often also with lower dimension.

Until recently, $W$-orbits were used as an efficient computational tool, particularly for large-scale computations (see for example [2, 6, 15, 16] and references therein). Their appreciation as point sets defining families of $W$-invariant special functions of $n$ variables is relatively recent [10, 11, 19]. Other possible applications could include an unusual twist of some symmetry breaking problems in physics, where, rather than breaking down weight systems of representations, one would break each orbit independently.

The main advantage of the projection matrices method is the uniformity of its application as to the different algebra–subalgebra pairs, which makes it particularly amenable to computer implementation. Thus in [20], branching rules for representations of dimension up to 5000 were calculated for all simple Lie algebras of rank up to 8 and for all their maximal semisimple subalgebras. Corresponding projection matrices were presented as a computational tool only later in [14]. Subsequently, the tables [12] were also based on their exploitation.

Particular Weyl group orbit reduction has undoubtedly been addressed on many occasions in the literature. As a separate subject of interest, orbit branching rules seem to have been first found in [13], where they are used for the reduction of many representations as well as orbits of the five exceptional simple Lie algebras. The corresponding projection matrices are shown there too. In [5], several generating functions for the reduction problem were derived. It is a very efficient method, in that it solves the problem for all orbits at once. Unfortunately, for each algebra–subalgebra pair, a new generating function needs to be derived. An independent original approach to orbit–orbit branching rules can be found in [22, 23], in which essentially combinatorial algorithms are developed for specific series of algebra–subalgebra pairs. For $A_n$, an algorithm for the equal rank subalgebra series of cases can be found there. It should be compared with subsection 4.3.

Our problem in this paper is closely related to the computation of branching rules for irreducible finite-dimensional representations of simple Lie algebras (equivalently, to branching rules for weight systems of representations). Theoretically, such problems need to be solved while describing symmetry breaking in some physical systems. Practically, the orbit branching rules problem needs to be solved whenever a large-scale computation of branching rules for representations is undertaken. The similarity of the two problems is in the transformation of orbit points (weights) that takes place in both cases. However, there are practically important differences between the two problems. The orbit branching rules are less constrained than those for representations. Some of the differences were already pointed out in [7]. Here we underline just two.

(i) While weight systems grow without limits, the larger the representation one has to work with, the orbit size (the number of points in an orbit) is always bounded by the order of the corresponding Weyl group. Without limits, only the distance of the orbit points from the origin can grow, but not their number. A weight system of a representation is a union of several $W$-orbits. The higher the representation, the more orbits it comprises. In general, to determine the orbits that form the weight system of a representation (equivalently, to compute dominant weights multiplicities in a representation) is often a difficult and laborious task (see [2] and references therein). Therefore, any large-scale computation with representations practically imposes the need to break a large problem for the weight system into a series of much smaller ones for individual orbits. The computation of branching rules for the representations is one such problem; the decomposition of products of representations into the direct sum of irreducible representations is another problem, which often needs to be carried out for relatively large representations, and which is solved entirely using the weight systems, see for example [6].

(ii) A point of a weight system of a representation necessarily belongs to a weight lattice of the Lie algebra. Its coordinates are integers in a suitably chosen basis of $\mathbb{R}^n$, so are the points of orbits after reduction. When we work with an individual orbit, we are free to choose the orbit, that is, the seed point $\lambda$, anywhere in $\mathbb{R}^n$, as close or as distant from the origin or from any other lattice point as one desires. After the reduction, some orbits can be very close, while some are far apart. Examples of such effects are shown in the concluding remarks of [7]. The flexibility thus achieved needs yet to be exploited.

The branching rules for $W(A_n) \to W(L)$, where $L$ is a maximal reductive subalgebra of $A_n$, is a linear transformation between Euclidean spaces $\mathbb{R}^n \to \mathbb{R}^m$, where $m$ is the rank of $L$. The branching rules are unique, unlike transformations of individual orbit points, which depend on the relative choice of bases. In this paper, we provide the linear transformation in the form of an $n \times m$ matrix, the 'projection matrix'. A suitable choice of bases allows one to obtain integer matrix elements in all the projection matrices listed here. Note that we use Dynkin notations and numberings for roots, weights and diagrams.

## 2. Preliminaries

The general strategy of our approach can be described as follows.

Consider the pair $L \supset L'$ of Lie algebras of ranks $n$ and $m$, respectively, where $L$ is simple and $L'$ is maximal reductive. In principle, the orbit reduction problem for the pair

$W(L) \supset W(L')$ is solved when the $n \times m$ matrix $P$ is found, with the property that points of any orbit of $W(L)$ are transformed/projected by $P$ into points of the corresponding orbits of $W(L')$. The computation of the branching rule for a specific orbit amounts to applying $P$ to the points of the orbit, and to sorting out the projected points according to the orbits of $W(L')$.

This task requires that one be able to calculate the points of any orbit of the Weyl group of any semisimple Lie algebra encountered here. There is a standard method, but we refrain from describing it here once again. Instead we refer to [7], the immediate predecessor of this paper, wherein all orbit points are given relative to the so-called $\omega$-basis. Geometric relations between the basis vectors are described by the matrix $(\langle\omega_j, \omega_k\rangle)$ of scalar products of the basis vectors. The matrices are found in [2] under the name 'quadratic form matrices' for all simple Lie algebras.

The Weyl group of the one-parameter Lie algebra $U_1$ is trivial, consisting of the identity element only. This algebra is present in reductive non-semisimple Lie algebras. Its irreducible representations are all one dimensional; hence, its orbits consist of one element. They are labeled by integers. The symbol $(k)$ may stand for either the orbit $\{k, -k\}$ of $A_1$, or for the $U_1$-orbit of one point $\{k\}$. Distinction should be made from the context. For example, the orbit $(p)(q)$, where $p \in \mathbb{Z}^{>0}$, $q \in \mathbb{Z}$, of $W(A_1 \times U_1)$ has two elements, $\{(p)(q), (-p)(q)\}$.

All orbits of $W(A_n)$ have the following symmetry. For each point $(a_1, a_2, \ldots, a_n)$ that belongs to an orbit, the point $(-a_n, -a_{n-1}, \ldots, -a_2, -a_1)$ also belongs to the same orbit. We say that the orbits of $W(A_n)$ in the following pair are contragredient:

$$(q_1, q_2, \ldots, q_{n-1}, q_n), \qquad (q_n, q_{n-1}, \ldots, q_2, q_1), \qquad q_j \geqslant 0 \qquad \text{for all } j.$$

Branching rules for contragredient orbits are closely related. They either coincide, or one can be obtained from the other by interchanging $q_k \leftrightarrow q_{n-k}$ components of the dominant points. We list only one such pair of branching rules.

It is known that the fundamental representations, i.e. representations with highest weight equal to $\omega_j$, $j = 1, \ldots, n$, have weight systems consisting only of the one Weyl group orbit $W_{\omega_j}$. If no other orbits are involved in a branching rule, that rule coincides with the branching rule for representations.

The number of points in a Weyl group orbit, labeled by its unique dominant weight $(a_1, a_2, \ldots, a_n)$, is determined by the $a_j$'s that are strictly positive. In orbits encountered in representation theory, we have $a_j \in \mathbb{Z}^{\geqslant 0}$. Since we are considering a more general setup, we require only $a_j \in \mathbb{R}^{\geqslant 0}$. If all $a_j$'s are strictly positive, the orbit of $W(A_n)$ contains $(n + 1)!$ points.

For simplicity of notation we subsequently identify cases by algebra–subalgebra symbols rather than by corresponding Weyl groups. In particular, we speak of an orbit of $A_k$ rather than of an orbit of $W(A_k)$.

## 3. Construction of projection matrices

The projection matrix $P$ for a given pair $L \supset L'$ of Lie algebras is calculated from one known branching rule. The classification of subalgebras amounts precisely to providing that branching rule. Usually the branching rule is given for the lowest dimensional representation. Then the matrix is obtained using the weight systems of the involved representations.

First, make a suitable (lexicographical) ordering of the weights of $L$ and $L'$. Then associate the weights on both sides one-by-one according to the chosen order. The matrix is obtained from requiring that each weight of $L$ be transformed to its associate weight of $L'$.

**Example 1.** Consider the case of $A_3 \supset C_2$ of subsection 5.2. The lowest orbit of $A_3$ contains four points. The lowest orbit of $C_2$ also contains four points. More precisely, there are two four-point orbits of $A_3$ and two such orbits of $C_2$. Either of the two $A_3$ orbits can be used for setting up the projection matrix. The two orbits of $C_2$ with dominant weights $(1, 0)$ and $(0, 1)$ are different, being related to simple roots of different length. We take the $A_3$ orbit of the dominant point $(1, 0, 0)$ and project it onto the $C_2$ orbit of the point $(1, 0)$. (See the second option in the last item of section 6 below.)

$$(1,0,0) \mapsto (1,0), \qquad (-1,1,0) \mapsto (-1,1),$$
$$(0,-1,1) \mapsto (1,-1), \qquad (0,0,-1) \mapsto (-1,0).$$

Writing the points as column matrices, the projection matrix of subsection 5.2 is obtained from the first three. Proceeding one column at a time, we have

$$\begin{pmatrix}1 & * & *\\ 0 & * & *\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}, \quad \begin{pmatrix}1 & 0 & *\\ 0 & 1 & *\end{pmatrix}\begin{pmatrix}-1\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}, \quad \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}0\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}.$$

Here, stars denote the entries that are still to be determined. The matrix $P = \left(\begin{smallmatrix}1 & 0 & 1\\ 0 & 1 & 0\end{smallmatrix}\right)$ then automatically transforms the fourth point $(0, 0, -1)$ of the $A_3$ orbit as required. This matrix can be used for projecting points of any $A_3$ orbit.

## 4. Equidimensional orbit branching rules

All orbits $W(A_n)$ are $n$-dimensional except for the trivial one $\lambda = 0$, which consists of one point, the origin. Points can be seen as vertices of a polytope in $\mathbb{R}^n$ [7]. Reduction to orbits of the same dimension occurs when reduced orbits have the symmetry of $W(A_r \times A_s \times U_1)$, where $r + s + 1 = n$. Clearly, we need to consider only the cases $r \geqslant s$. Geometrically, the orbit points are not displaced in this case; rather, they are relabeled by the coordinates given in the standard basis of the subgroup.

In this section, we first consider the lowest special cases in part as transparent illustration and in part because they are most frequently encountered in physics applications. Lastly, we consider the infinite series of cases $1 \leqslant n < \infty$ for all possible values of rank $n$: $W(A_n) \to W(A_{n-k-1} \times A_k \times U_1)$, $0 \leqslant k \leqslant \left[\frac{n-1}{2}\right]$, where $\left[\frac{n-1}{2}\right]$ is the integer part of $\frac{n-1}{2}$.

### *4.1. Orbit branching rules for $A_n \supset A_{n-1} \times U_1$*

*4.1.1.* $A_1 \supset U_1$. The lowest example is trivial. The Weyl group of $A_1$ has two elements; the Weyl group of $U_1$ is just the identity transformation. An orbit $\{p, -p\}$ of $A_1$ reduces to two orbits of $U_1$:

$$(p) \supset (p) + (-p), \qquad p \in \mathbb{R}^{>0}.$$

The reduction is accomplished by applying the $1 \times 1$ projection matrix $P = (1)$ to each element of the $A_1$ orbit.

*4.1.2.* $A_2 \supset A_1 \times U_1$. The second lowest example is often used in nuclear and particle physics. In terms of compact Lie groups it is $SU(3) \supset U(2) = SU(2) \times U(1)$. The reduction is accomplished by applying to each element of the $A_2$ orbit the projection matrix $P = \left(\begin{smallmatrix}1 & 0\\ 1 & 2\end{smallmatrix}\right)$, and by subsequently regrouping the results into orbits of $A_1 \times U_1$. We find the branching rules for the three types of $A_2$ orbits:

$$
\begin{aligned}
(p,0) &\supset (p)(p) + (0)(-2p),\\
(0,q) &\supset (q)(-q) + (0)(2q),\\
(p,q) &\supset (p)(p+2q) + (p+q)(p-q) + (q)(-2p-q),
\end{aligned}
$$

where $p, q \in \mathbb{R}^{>0}$.

*4.1.3.* $A_3 \supset A_2 \times U_1$. Reduction is achieved by applying to each element of an $A_3$ orbit the projection matrix

$$
\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 2 & 3 \end{pmatrix} \tag{1}
$$

and by subsequently regrouping the results into orbits of $A_2 \times U_1$. For all seven types of $A_3$ orbits, we find the branching rules:

$$
\begin{aligned}
&(p,0,0) \supset (p,0)(p) + (0,0)(-3p),\\
&(0,q,0) \supset (0,q)(2q) + (q,0)(-2q),\\
&(0,0,r) \supset (0,0)(3r) + (0,r)(-r),\\
&(p,q,0) \supset (p,q)(p+2q) + (p+q,0)(p-2q) + (q,0)(-3p-2q),\\
&(p,0,r) \supset (p,0)(p+3r) + (p,r)(p-r) + (0,r)(-3p-r),\\
&(0,q,r) \supset (0,q)(2q+3r) + (0,q+r)(2q-r) + (q,r)(-2q-r),\\
&(p,q,r) \supset (p,q)(p+2q+3r) + (p,q+r)(p+2q-r) + (p+q,r)(p-2q-r)\\
&\qquad\qquad + (q,r)(-3p-2q-r),
\end{aligned} \tag{2}
$$

where $p, q, r \in \mathbb{R}^{>0}$.

**Example 2.** Let us illustrate the actual computation of branching rules on the example of $A_3$ orbit (2, 0, 1) containing 12 points. We write the coordinates of the points as column vectors:

$$
\begin{gathered}
\begin{pmatrix}2\\0\\1\end{pmatrix}, \quad \begin{pmatrix}-2\\2\\1\end{pmatrix}, \quad \begin{pmatrix}2\\1\\-1\end{pmatrix}, \quad \begin{pmatrix}0\\-2\\3\end{pmatrix}, \quad \begin{pmatrix}-2\\3\\-1\end{pmatrix}, \quad \begin{pmatrix}3\\-1\\0\end{pmatrix},\\
\begin{pmatrix}0\\1\\-3\end{pmatrix}, \quad \begin{pmatrix}1\\-3\\2\end{pmatrix}, \quad \begin{pmatrix}-3\\2\\0\end{pmatrix}, \quad \begin{pmatrix}1\\-1\\-2\end{pmatrix}, \quad \begin{pmatrix}-1\\-2\\2\end{pmatrix}, \quad \begin{pmatrix}-1\\0\\-2\end{pmatrix}.
\end{gathered} \tag{3}
$$

Multiplying each of the points of (3) by the matrix (1), one gets the points of the $A_2 \times U_1$ orbits written as column vectors. Rewriting them in the horizontal form and remembering that the first two coordinates belong to $A_2$ orbits, the third one belonging to $U_1$, we have the set of projected points. It remains to distribute the points into individual orbits. Practically it suffices to select just the dominant ones (no negative coordinates) because they represent the orbits that are present. Results are given by (2), where $p = 2$ and $r = 1$.

*4.1.4. $A_n \supset A_{n-1} \times U_1, n \geqslant 2$.* The cases listed in subsections 4.1.2 and 4.1.3 are special cases of the present one:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 & \cdots & n-2 & n-1 & n \end{pmatrix}$$

We write here just a few branching rules for this case:

$$\begin{aligned}
&(p, 0, 0, \ldots, 0) \supset (p, 0, 0, \ldots, 0)(p) + (0, \ldots, 0)(-np), \\
&(0, q, 0, \ldots, 0) \supset (0, q, 0, \ldots, 0)(2q) + (q, 0, 0, \ldots, 0)(-(n-1)q), \\
&(p, 0, \ldots, 0, r) \supset (p, 0, 0, \ldots, 0)(p + nr) \\
&\qquad\qquad + (p, 0, 0, \ldots, 0, r)(p - r) + (0, 0, \ldots, 0, r)(-np - r).
\end{aligned}$$

Note that, here and everywhere below, $p, q, r \in \mathbb{R}^{>0}$.

### *4.2. Orbit branching rules for $A_n \supset A_{n-k-1} \times A_k \times U_1$*

All the cases so far can be viewed as the special cases of the present one where k = 0. Here we are considering the cases with general rank $n \geqslant 3$ and $1 \leqslant k \leqslant [\frac{n-1}{2}]$.

*4.2.1. $A_3 \supset A_1 \times A_1 \times U_1$.* The reduction is accomplished by applying to each element of the $A_3$ orbit the projection matrix $P = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{smallmatrix}\right)$, and by subsequently regrouping the results into orbits of $A_1 \times A_1 \times U_1$.

For all types of $A_3$ orbits, we find

$$\begin{aligned}
&(p, 0, 0) \supset (p)(0)(p) + (0)(p)(-p), \\
&(0, q, 0) \supset (0)(0)(2q) + (0)(0)(-2q) + (q)(q)(0), \\
&(0, 0, r) \supset (0)(r)(r) + (r)(0)(-r), \\
&(p, q, 0) \supset (p)(0)(p + 2q) + (p + q)(q)(p) + (q)(p + q)(-p) + (0)(p)(-p-2q), \\
&(p, 0, r) \supset (p)(r)(p + r) + (p + r)(0)(p - r) + (0)(p + r)(r-p) + (r)(p)(-p - r), \\
&(0, q, r) \supset (0)(r)(2q + r) + (q)(q + r)(r) + (q + r)(q)(-r) + (r)(0)(-2q-r), \\
&(p, q, r) \supset (p)(r)(p + 2q + r) + (p + q)(r + q)(p + r) + (p + q + r)(q)(p - r) \\
&\qquad + (q)(p + q + r)(r-p) + (q + r)(p + q)(-p - r) + (r)(p)(-p-2q-r).
\end{aligned}$$

*4.2.2. $A_4 \supset A_2 \times A_1 \times U_1$.* In terms of compact Lie groups, this is the case frequently used in particle physics, namely $SU(5) \supset SU(3) \times SU(2) \times U(1)$. The reduction is accomplished by applying to each element of the $A_4$ orbit the projection matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 6 & 3 \end{pmatrix}$$

and by subsequently regrouping the results into orbits of $A_2 \times A_1 \times U_1$. For the following types of $A_4$ orbits, we find

$$
\begin{aligned}
&(p,0,0,0) \supset (p,0)(0)(2p) + (0,0)(p)(-3p),\\
&(0,q,0,0) \supset (0,q)(0)(4q) + (q,0)(q)(-q) + (0,0)(0)(-6q),\\
&(p,0,0,r) \supset (p,0)(r)(2p+3r) + (p,r)(0)(2p-2r)\\
&\qquad\qquad + (0,0)(p+r)(3r-3p) + (0,r)(p)(-3p-2r).
\end{aligned}
$$

*4.2.3.* $A_n \supset A_{n-2} \times A_1 \times U_1$, *for odd* $n \geqslant 3$. The projection matrix is

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1\\
1 & 2 & 3 & 4 & \cdots & n-3 & n-2 & n-1 & \frac{n-1}{2}
\end{pmatrix},
$$

and some of the branching rules are

$$
\begin{aligned}
&(p,0,0,\ldots,0) \supset (p,0,\ldots,0)(0)(p) + (0,\ldots,0)(p)\left(-\frac{n-1}{2}p\right),\\
&(0,q,0,\ldots,0) \supset (0,q,0,\ldots,0)(0)(2q) + (q,0,0,\ldots,0)(q)\left(-\frac{n-3}{2}q\right)\\
&\qquad\qquad + (0,\ldots,0)(0)((1-n)q),\\
&(p,0,\ldots,0,r) \supset (p,0,\ldots,0)(r)\left(p+\frac{n-1}{2}r\right) + (p,0,\ldots,0,r)(0)(p-r)\\
&\qquad\qquad + (0,\ldots,0)(p+r)\left((r-p)\frac{n-1}{2}\right) + (0,\ldots,0,r)(p)\left(-r-\frac{n-1}{2}p\right).
\end{aligned}
$$

*4.2.4.* $A_n \supset A_{n-2} \times A_1 \times U_1$, *for even* $n \geqslant 4$. The projection matrix is

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1\\
2 & 4 & 6 & 8 & \cdots & 2(n-3) & 2(n-2) & 2(n-1) & n-1
\end{pmatrix},
$$

and some of the branching rules are

$$
\begin{aligned}
&(p,0,0,\ldots,0) \supset (p,0,\ldots,0)(0)(2p) + (0,\ldots,0)(p)((1-n)p),\\
&(0,q,0,\ldots,0) \supset (0,q,0,\ldots,0)(0)(4q) + (q,0,\ldots,0)(q)((3-n)q)\\
&\qquad\qquad + (0,\ldots,0)(0)(2(1-n)q),\\
&(p,0,\ldots,0,r) \supset (p,0,\ldots,0)(r)(2p+(n-1)r) + (p,0,\ldots,0,r)(0)(2(p-r))\\
&\qquad\qquad + (0,\ldots,0)(p+r)((n-1)(r-p)) + (0,\ldots,0,r)(p)(-2r-(n-1)p).
\end{aligned}
$$

### 4.3. *The general case* $A_n \supset A_{n-k-1} \times A_k \times U_1 : 1 \leqslant k \leqslant [\frac{n-1}{2}]$

The branching rules of subsection 4.2 are important special cases of the general case. The projection matrix in the general case can be written as

$$\left(\begin{array}{c|c|c} I_{n-k-1} & \multicolumn{2}{c}{\mathbf{0}} \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & I_k \\ \hline \underbrace{k{+}1\ 2(k{+}1)\cdots(n{-}k{-}1)(k{+}1)}_{n-k-1} & \underbrace{(n{-}k)(k{+}1)}_{1} & \underbrace{k(n{-}k)\cdots 2(n{-}k)\ n{-}k}_{k} \end{array}\right) \begin{array}{l} \}\, n-k-1 \\ \}\, k \\ \}\, 1 \end{array}$$

Note that, here and everywhere below, $I_k$ denotes the $k \times k$ identity matrix and $\mathbf{0}$ represents the zero matrix:

$$\begin{aligned}
&(p, 0, 0, \ldots, 0) \supset (p, 0, \ldots, 0)(0, \ldots, 0)((k+1)p) + (0, \ldots, 0)(p, 0, \ldots, 0)((k-n)p),\\
&(p, 0, \ldots, 0, r) \supset (p, 0, \ldots, 0)(0, \ldots, 0, r)((k+1)p + (n-k)r)\\
&\qquad + (p, 0, \ldots, 0, r)(0, \ldots, 0)((k+1)(p-r))\\
&\qquad + (0, \ldots, 0)(p, 0, \ldots, 0, r)((n-k)(r-p))\\
&\qquad + (0, \ldots, 0, r)(p, 0, \ldots, 0)((-k-1)r + (k-n)p).
\end{aligned}$$

## 5. Branching rules for maximal semisimple subalgebras of $A_n$

The simple Lie algebras $A_n$ contain no semisimple subalgebras of the same rank $n$. Hence all orbit branching rules considered in this section have rank strictly smaller than $n$. We proceed by increasing rank values until $n = 8$. Then we describe the infinite series involving the Weyl groups of classical Lie algebras, namely $W(A_{2n}) \supset W(B_n)$, $n \geqslant 3$, $W(A_{2n-1}) \supset W(C_n)$, $n \geqslant 2$, and $W(A_{2n-1}) \supset W(D_n)$, $n \geqslant 4$.

We include the low-rank special cases of the three infinite series. We exclude the cases when a subalgebra is maximal among semisimple Lie algebras, but not among reductive algebras. Projection matrices for the latter cases are obtained by striking the last line of the corresponding matrices from the previous section. Dots in the projection matrices denote zero matrix elements.

### 5.1. *Rank 2*

There is only one case here, namely $A_2 \supset A_1$, which is often specified in terms of corresponding Lie groups either as $SU(3) \supset O(3)$, if the groups should be compact, or as $Sl(3, \mathbb{C}) \supset O(3, \mathbb{C})$, if the groups have complex parameters. Their Weyl group orbits are the same. The projection matrix is $P = (2\ 2)$, so that we obtain the reductions

$$(p, q) \supset (2p + 2q) + (2p) + (2q), \qquad (p, 0) \supset (2p) + (0), \qquad (0, q) \supset (2q) + (0). \tag{4}$$

**Example 3.** Let us underline the geometrical content of relations (4). On the left-hand side, there are points in $\mathbb{R}^2$ given by their coordinates in $\omega$-basis $\{\omega_1, \ \omega_2\}$ of $A_2$. The geometric relation of the two basis vectors is given by the $2 \times 2$ matrix of scalar products $\langle\omega_j, \omega_k\rangle$. In $A_n$, it happens to be the inverse $C^{-1}$ of the Cartan matrix of the algebra. In particular, for $A_2$, we have $C^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. It follows that the basis vectors are of equal length $\sqrt{2/3}$, and that $\angle(\omega_1, \omega_2) = 60°$.

On the right-hand side of (4), there are the $A_1$ orbit points in $\mathbb{R}^1 \subset \mathbb{R}^2$. Applying to $A_1$ the same rules as previously applied to $A_2$, we have $C = (2)$ so that $C^{-1} = (1/2)$. Thus, the basis vector of $A_1$, say $\omega$, has the length $1/\sqrt{2}$.

It remains to clarify what are the relative positions of $\omega_1$, $\omega_2$ and $\omega$. The theory leaves us several options. A reasonable choice is built-in into the construction of the projection matrix in each case. Justification for this is outside the scope of this paper. For additional information, see [14]. However, the relative positions of basis vectors in $\mathbb{R}^2$ and $\mathbb{R}^1$ are established, for example, from

$$P\omega_1 = (2\ 2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = P\omega_2 = 2\omega.$$

Since equal-length vectors $\omega_1$ and $\omega_2$ are projected into the same point on the $\omega$-axis, the direction of $\omega$ divides the angle between $\omega_1$ and $\omega_2$ into equal parts.

### 5.2. Rank 3

There are just two cases to consider. We write only their projection matrices.

$$A_3 \supset C_2 : \quad \begin{pmatrix} 1 & \cdot & 1 \\ \cdot & 1 & \cdot \end{pmatrix}, \qquad A_3 \supset A_1 \times A_1 : \quad \begin{pmatrix} 1 & \cdot & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

**Example 4.** There are 12 points in (3). Let us transform them by the matrix $\begin{pmatrix} 1 & \cdot & 1 \\ \cdot & 1 & \cdot \end{pmatrix}$. Two dominant points are found when writing the projected points in horizontal form, namely (3, 0) and (1, 1). Hence we have the $A_3 \supset C_2$ rule $(2, 0, 1) \supset (3, 0) + (1, 1)$. The orbit (3, 0) contains four points and the orbit (1, 1) contains eight points.

Geometrically, (2, 0, 1) is a tetrahedron with four cut-off vertices. The planar figure after the projection is the union of the square (3, 0) and the octagon (1, 1).

Let us underline the difference between the subalgebra $A_1 \times A_1$ here and the one in subsection 4.2.1. Using the corresponding projection matrices, we obtain respectively the reductions

$$(1, 0, 0) \supset (1)(1), \quad \text{and} \quad (1, 0, 0) \supset (1)(0)(1) + (0)(1)(-1).$$

Ignoring the contribution from $U_1$ in the second branching rule, the four orbit points obtained after the reduction are different in the two cases:

$$(1, 0, 0) \supset \{(1)(1), (-1)(1), (1)(-1), (-1)(-1)\},$$
$$(1, 0, 0) \supset \{(1)(0), (-1)(0), (0)(1), (0)(-1)\}.$$

There is an obvious subalgebra $A_2$ in $A_3$. Although it is maximal among semisimple subalgebras of $A_3$, it is not maximal among reductive subalgebras. It coincides with $A_2$ in subsection 4.1.3.

### *5.3. Rank 4*

There is only one simple and maximal subalgebra of $A_4$ among the reductive subalgebras:

$$A_4 \supset C_2 : \quad \begin{pmatrix} \cdot & 2 & 2 & \cdot \\ 1 & \cdot & \cdot & 1 \end{pmatrix}.$$

The other two semisimple subalgebras of rank 3 of $A_4$, namely $A_3$ and $A_1 \times A_2$, can be both extended by $U_1$ to maximal reductive subalgebras. They are the special cases $n = 4$ found in subsections 4.1.4 and 4.2.4 respectively.

Some branching rules:

$$\begin{aligned} &(p,0,0,0) \supset (0,p) + (0,0), \\ &(p,0,0,r) \supset (0,p+r) + (0,p) + (0,r) + (2r,p-r), \qquad p > r, \\ &(p,0,0,p) \supset (0,2p) + 2(0,p) + 2(2p,0). \end{aligned}$$

### *5.4. Rank 5*

There are four maximal subalgebras in this case. The first two are special cases of the general inclusions of subsection 5.8. The Lie algebras $A_3$ and $D_3$ coincide, except that by general convention we agreed not to consider the $D_3$ form. Therefore, $A_5 \supset A_3$ can be read equivalently as $A_5 \supset D_3$, provided that we modify the order of point coordinates as follows: $(a, b, c)$ of $A_3$ corresponds to $(b, a, c)$ of $D_3$:

$$A_5 \supset A_3 : \quad \begin{pmatrix} \cdot & 1 & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & 2 & 1 & \cdot \end{pmatrix}, \qquad A_5 \supset C_3 : \quad \begin{pmatrix} 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix},$$

$$A_5 \supset A_2 : \quad \begin{pmatrix} \cdot & 1 & 3 & 2 & 2 \\ 2 & 2 & \cdot & 1 & \cdot \end{pmatrix}, \qquad A_5 \supset A_1 \times A_2 : \quad \begin{pmatrix} 1 & \cdot & 1 & \cdot & 1 \\ 1 & 2 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 2 & 1 \end{pmatrix}.$$

In particular, the branching rules for the $A_5$ orbit of six points are

$$(p,0,0,0,0) \supset \begin{cases} (0,p,0) & \text{for} \quad A_3 \\ (p,0,0) & \text{for} \quad C_3 \\ (0,2p) & \text{for} \quad C_2 \\ (p)(p,0) & \text{for} \quad A_1 \times A_2 \end{cases} \qquad p \in \mathbb{R}^{>0}. \tag{5}$$

The first two are special cases of (8) and (7), respectively.

### *5.5. Rank 6*

The only entry here is a special case of $A_{2n} \supset B_n$ of subsection 5.8, and its branching rules:

$$A_6 \supset B_3 : \quad \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 2 & 2 & \cdot & \cdot \end{pmatrix}.$$

$$\begin{aligned} &(p,0,0,0,0,0) \supset (p,0,0) + (0,0,0), \\ &(p,0,0,0,0,r) \supset (p+r,0,0) + (p,0,0) + (r,0,0) + (p-r,r,0), \qquad p > r, \\ &(p,0,0,0,0,p) \supset (2p,0,0) + 2(p,0,0) + 2(0,p,0). \end{aligned}$$

### 5.6. Rank 7

The first two of the three cases are restrictions to $n = 7$ of the corresponding general cases of subsection 5.8:

$$A_7 \supset C_4 : \quad \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}, \qquad A_7 \supset D_4 : \quad \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 2 & 1 & \cdot & \cdot \end{pmatrix},$$

$$A_7 \supset A_1 \times A_3 : \quad \begin{pmatrix} 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 \\ 1 & 2 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 2 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 2 & 1 \end{pmatrix}.$$

In particular, for $A_7 \supset A_1 \times A_3$, we obtain

$$\begin{aligned}
&(p, 0, 0, 0, 0, 0, 0) \supset (p)(p, 0, 0),\\
&(p, 0, 0, 0, 0, 0, r) \supset (p+r)(p, 0, r) + (p-r)(p, 0, r) + (p+r)(p-r, 0, 0), \qquad p > r,\\
&(p, 0, 0, 0, 0, 0, p) \supset (2p)(p, 0, p) + 2(0)(p, 0, p) + 4(2p)(0, 0, 0).
\end{aligned}$$

### 5.7. Rank 8

The first case is a special case of (6):

$$A_8 \supset B_4 : \quad \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & 2 & \cdot & \cdot & \cdot \end{pmatrix},$$

$$A_8 \supset A_2 \times A_2 : \quad \begin{pmatrix} 1 & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot \\ \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot & 1 \\ 1 & 2 & 1 & 2 & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & 2 & 1 & 2 & 1 \end{pmatrix}.$$

Examples of the branching rules for the second case:

$$\begin{aligned}
&(p, 0, 0, 0, 0, 0, 0, 0) \supset (p, 0)(p, 0),\\
&(p, 0, 0, 0, 0, 0, 0, r) \supset (p, r)(p, r) + (p-r, 0)(p, r) + (p, r)(p-r, 0), \qquad p > r,\\
&(p, 0, 0, 0, 0, 0, 0, p) \supset (p, p)(p, p) + 3(0, 0)(p, p) + 3(p, p)(0, 0).
\end{aligned}$$

### 5.8. Three general rank cases

The cases are presented with examples of branching rules for the orbits $(p, 0, \ldots, 0)$ and $(p, 0, \ldots, 0, r)$, where the parameters $p,\ r$ are strictly positive and real. We also assume that $p > r$. If $p < r$ the parameters $p$ and $r$ in the branching rule need to be interchanged. The case $p = r$ often needs to be listed separately.

$A_{2n} \supset B_n, n \geqslant 3$

$$P = \left(\begin{array}{c|c|c} I_{n-1} & \mathbf{0} & E_{n-1} \\ \hline 0 \cdots 0 & 2\,2 & 0 \cdots 0 \end{array}\right), \tag{6}$$

Note that, here and everywhere below, $E_k$ denotes the $k \times k$ matrix with units on the codiagonal:

$(p, 0, 0, \ldots, 0) \supset (p, 0, \ldots, 0) + (0, \ldots, 0),$
$(p, 0, \ldots, 0, r) \supset (p + r, 0, \ldots, 0) + (p, 0, \ldots, 0) + (r, 0, \ldots, 0) + (p - r, r, 0, \ldots, 0),$
$(p, 0, \ldots, 0, p) \supset (2p, 0, \ldots, 0) + 2(p, 0, \ldots, 0) + 2(0, p, 0, \ldots, 0).$

$A_{2n-1} \supset C_n, n \geqslant 2$

$$P = \left(\begin{array}{c|c|c} I_{n-1} & \mathbf{0} & E_{n-1} \\ \hline 0 \cdots 0 & 1 & 0 \cdots 0 \end{array}\right) \tag{7}$$

$$\begin{aligned} &(p, 0, 0, \ldots, 0) \supset (p, 0, \ldots, 0), \\ &(p, 0, \ldots, 0, r) \supset (p + r, 0, \ldots, 0) + (p - r, r, 0, \ldots, 0), \\ &(p, 0, \ldots, 0, p) \supset (2p, 0, \ldots, 0) + 2(0, p, 0, \ldots, 0). \end{aligned}$$

$A_{2n-1} \supset D_n, n \geqslant 4$

$$P = \left(\begin{array}{c|c|c} I_{n-1} & \mathbf{0} & E_{n-1} \\ \hline 0 \cdots 0\ 1 & 2 & 1\ 0 \cdots 0 \end{array}\right) \tag{8}$$

$$\begin{aligned} &(p, 0, 0, \ldots, 0) \supset (p, 0, \ldots, 0), \\ &(p, 0, \ldots, 0, r) \supset (p + r, 0, \ldots, 0) + (p - r, r, 0, \ldots, 0), \\ &(p, 0, \ldots, 0, p) \supset (2p, 0, \ldots, 0) + 2(0, p, 0, \ldots, 0). \end{aligned}$$

## 6. Concluding remarks

- The pairs $W(L) \supset W(L')$ in this paper involve a maximal subalgebra $L'$ in $L$. A chain of maximal subalgebras linking $L$ and any of its reductive non-maximal subalgebras $L''$ can be found. Corresponding projection matrices combine, by the common matrix multiplication, into the projection matrix for $W(L) \supset W(L'')$.
- Projection matrices of section 4 are square matrices with determinant different from zero. Hence they can be inverted and used in the opposite direction. The inverse matrix transforms an orbit of $W(L')$ into the linear combination of orbits of $W(L)$, where $L' \subset L$. The linear combination has integer coefficients of both signs in general. We know of no interpretation of such 'branching rules' in the applied literature, although they have their place in the Grothendieck rings of representations.
- Curious and completely unexplored relations between pairs of maximal subalgebras, say $L'$ and $L''$, of the same Lie algebra $L$ can be found by combining the projection matrices $P(L \to L')$ and $P(L \to L'')$ as
$$P(L' \to L'') = P(L \to L'')P^{-1}(L \to L').$$
- The index of a semisimple subalgebra in a simple Lie algebra is an invariant of all branching rules for a fixed algebra–subalgebra pair. It was introduced in [4], see equation (2.26). It is an invariant also for any pair $W(L) \supset W(L')$.
- Congruence classes of representations are naturally extended to congruence classes of $W$-orbits [7]. Comparing the congruence classes of orbits for $W(L) \supset W(L')$ reveals that not all combinations of congruence classes are present. A relative congruence class is a valid and useful concept which deserves investigation.

- Here, the relations between orbits were defined by the classification of maximal reductive subalgebras in simple type-$A_n$ Lie algebras. There exists another relation between such algebras that is not a homomorphism. It is called subjoining [17, 21]. Consider an example. The four-dimensional representation $(1, 0, 0)$ of $A_3$ does *not* contain the five-dimensional representation $(0, 1)$ of $C_2$. In spite of that, the projection matrix that maps the highest weight orbit of $A_3$ to the orbit $(0, 1)$ of $C_2$ can be obtained. Indeed, that projection matrix is $\left(\begin{smallmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right)$. The classification of maximal subjoinings in simple Lie algebras is found in [17].

## Acknowledgments

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada and by the MIND Research Institute. We are grateful for the hospitality extended to MN at the Centre de recherches mathématiques, Université de Montréal, where the work was done.

## References

[1] Borel A and de Siebental J 1949 Les sous-groupes fermés de rang maximum de groupes de Lie clos *Comment. Math. Helv.* **23** 200–21

[2] Bremner M R, Moody R V and Patera J 1985 *Tables of Dominant Weight Multiplicities for Representations of Simple Lie Algebras* (New York: Dekker)

[3] Champagne B, Kjiri M, Patera J and Sharp R T 1995 Description of reflection generated polytopes using decorated Coxeter diagrams *Can. J. Phys.* **73** 566–84

[4] Dynkin E B 1957 Semisimple subalgebras of semisimple Lie algebras *AMS Trnanslations, Ser. 2* **6** 111–244

[5] Gingras F, Patera J and Sharp R T 1992 Orbit-orbit branching rules between simple low-rank algebras and equal-rank subalgebras *J. Math. Phys.* **33** 1618–26

[6] Grimm S and Patera J 1997 Decomposition of tensor products of the fundamental representations of $E_8$ *Advances in Mathematical Sciences—CRM's 25 Years* (*CRM Proc. Lecture Notes* vol 11) ed L Vinet (Providence, RI: American Mathematical Society) pp 329–55

[7] Háková L, Larouche M and Patera J 2008 The rings of $n$-dimensional polytopes *J. Phys. A: Math. Theor.* **41** 495202 (arXiv:0901.4686)

[8] Humphreys J E 1972 *Introduction to Lie Algebras and Representation Theory* (New York: Springer)

[9] Humphreys J E 1990 *Reflection Groups and Coxeter Groups* (Cambridge: Cambridge University Press)

[10] Klimyk A and Patera J 2006 Orbit functions *SIGMA* **2** 006 (60 pp) (arXiv:math-ph/0601037)

[11] Klimyk A and Patera J 2007 Antisymmetric orbit functions *SIGMA* **3** 023 (83 pp) (arXiv:math-ph/0702040)

[12] McKay W G and Patera J 1981 *Tables of Dimensions, Indices, and Branching Rules for Representations of Simple Lie Algebras* (New York: Dekker)

[13] McKay W G, Patera J and Rand D 1990 *Tables of Representations of Simple Lie Algebras, vol I: Exceptional Simple Lie Algebras* (Montréal: Les Publications CRM) p 318

[14] McKay W G, Patera J and Sankoff D 1977 The computation of branching rules for representations of semisimple Lie algebras *Computers in Nonassociative Rings and Algebras* ed J Beck and B Kolman (New York: Academic)

[15] Moody R V and Patera J 1984 Characters of elements of finite order in simple Lie groups *SIAM J. Algebr. Discrete Methods* **5** 359–83

[16] Moody R V and Patera J 1987 Computation of character decompositions of class functions on compact semisimple Lie groups *Math. Comput.* **48** 799–827

[17] Moody R V and Pianzola A 1983 $\lambda$-mappings of representation rings of Lie algebras *Can. J. Math.* **35** 898–960

[18] Nesterenko M, Patera J and Tereszkjewicz A 2009 Orbit functions of $SU(n)$ and Chebyshev polynomials arXiv:0905.2925

[19] Patera J 2005 Compact simple Lie groups and theirs $C$-, $S$-, and $E$-transforms *SIGMA* **1** 025 (6 pp) (arXiv:math-ph/0512029)

[20] Patera J and Sankoff D 1973 *Branching Rules for Representations of Simple Lie Algebras* (Montréal: Presses Université de Montréal) p 99

[21] Patera J, Sharp R T and Slansky R 1980 On a new relation between semisimple Lie algebras *J. Math. Phys.* **21** 2335–41

[22] Thoma M and Sharp R T 1996 Orbit-orbit branching rules between classical simple Lie algebras and maximal reductive subalgebras *J. Math. Phys.* **37** 6570–81

[23] Thoma M and Sharp R T 1996 Orbit-orbit branching rules for families of classical Lie algebra-subalgebra pairs *J. Math. Phys.* **37** 4750–7